Approximate Controllability of a Class of Semilinear Hilfer Fractional Differential Equations

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Results in Mathematics

Approximate Controllability of a Class of Semilinear Hilfer Fractional Differential Equations

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Abstract. This article studies the approximate controllability for a class of fractional control system with analytic semigroup governed by differential equations with Hilfer fractional derivatives of order $\delta \in (0,1)$ and type $\zeta \in [0,1]$ in a Banach space. The existence and uniqueness of the mild solution is established with the help of semigroup theory, fractional power of operators and a generalized contraction type fixed point theorem. Further, a set of sufficient conditions is formulated for the approximate controllability of the system under consideration. The result obtained holds irrespective of whether the generated semigroup is compact or non-compact.

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Keywords. Fractional differential equations, Hilfer derivative, analytic semigroup, mild solutions, fixed point theorem, approximate controllability.

1. Introduction

Differential equation involving fractional derivative has emerged as an important area of investigation and is considered to be of immense significance in many branches of science and engineering where the nonlocal condition plays a vital role. The terms local and nonlocal are distinguished as follows: in order to calculate integer order derivatives of a function, it is required to know its properties in an infinitesimal neighborhood of the considered point whereas the fractional derivative relies not only on the present state but also upon all

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of its past states. As a result, integer order derivatives cannot describe processes with memory and this fact acts as the primary advantage of fractional derivatives over classical derivatives. Fractional differential equation provides a powerful tool for modeling numerous real life dynamic processes as it can describe their behavior more accurately. One can find its applications in signal and image processing, atmospheric diffusion of pollution, transmission of ultrasound waves, cellular diffusion processes, feedback amplifiers, the effect of speculation on the profitability of stocks in financial markets, and many more. For more details on this topic, we refer the reader to [3,14,21] and the references therein.

There are many possible generalizations of the n-th order differential operator $\frac{d^n}{d\tau^n}$ to the case when n is not an integer, named as Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard's etc. These operators interpolate between integer order differential operators. The most popular ones among them are the derivatives expressed in the Riemann-Liouville and Caputo sense. In [8], Hilfer proposed a new definition of fractional derivative, called Hilfer fractional derivative (also known as generalized Riemann-Liouville derivative), which includes both Riemann's and Caputo's definitions as particular cases. Subsequently, many researchers have studied the existence and uniqueness of nonlinear evolution equations involving Hilfer fractional derivatives [5,6,26].

For finite dimensional system, the concept of controllability for linear systems was introduced by Kalman [11,12]. In 1975, Triggiani [22] extended the theory of controllability from finite dimension to linear systems in infinite dimensions under the assumption that the operator acting on the state is bounded. Controllability of a system means the property of being able to steer between two arbitrary points in the state space. There are mainly two basic concepts of controllability: exact controllability and approximate controllability. Exact controllability means that the system can be steered from an arbitrary initial state to a desired final state whereas approximate controllability steers the system to an arbitrarily small neighborhood of the final state. Therefore, approximate controllability is basically a weaker concept than exact controllability. For information on other types of controllability that are present in literature, we refer the reader to Chalishajar et al. [1] and the references therein.

There are various means to establish that a system is approximately controllable. In [17,19,25], the controllability Grammian and fixed point theorems were used. For the approximate controllability of the considered evolution systems, it is assumed that the corresponding linear system is approximately controllable and the nonlinear function is uniformly bounded. Zhou [28] used the sequential approach and obtained some sufficient conditions for the approximate controllability of an integer order semi-linear equation. Thereafter, several researchers, e.g., [15,16] etc., have used this approach to study the approximate controllability of nonlinear evolution equations using different fractional order derivatives.

Dauer and Mahmudov [2] considered the following semi-linear evolution equation with finite delay:

$$
\begin{aligned}
y(\tau) =& Q(\tau)\phi(0) + \int_0^\tau Q(\tau - s)\big[Bu(s) + f(s, y_s, u(s))\big]ds, \quad \tau \in (0, b],\\
y_0(\theta) =& \Theta(\theta), \quad \theta \in [-h, 0],
\end{aligned}
$$

where $y(.) \in Y$, which is a Hilbert space, $u(.) \in L^2([0, b], U)$, U a Hilbert space. $\{Q(\tau)\}_{\tau>0}$ is a compact linear semigroup on Y, $B\colon U \to Y$ is a bounded linear operator and $\Theta \in C([-h, 0], Y)$. For $y \in C([-h, b], Y)$ and each $\tau \in [0, b]$, y_τ is defined by $y_\tau(\theta) = y(\tau+\theta)$, for $\theta \in [-h, 0]$. Here the controllability Grammian and Schauder's fixed point theorem were used to obtain sufficient conditions for approximate controllability of the system.

Another method for showing the approximate controllability is the one which establishes to show an inclusion relation between the reachable sets of the considered system and a linear system, which is assumed to be approximately controllable. In [9,10], Jeong and others considered the approximate controllability of ordinary semi-linear differential systems and used the concept of Lebesgue point to show the existence of a control function which steered the solution of the system to an arbitrary ϵ neighborhood ($\epsilon > 0$) of the desired final state.

Sukavanam and Kumar [20] discussed the approximate controllability of the following semi-linear delay equation with Caputo fractional derivative:

$$
\begin{aligned}
&{}^C D^\delta y(\tau) = Ay(\tau) + Bu(\tau) + f(\tau, y_\tau, u(\tau)), \quad \tau \in (0, b],\ \delta \in \left(\frac{1}{2}, 1\right),\\
&y_0(\theta) = \Theta(\theta), \quad \theta \in [-h, 0],
\end{aligned}
$$

where A is the generator of a C_0-semigroup, Y and U are Banach spaces, the state $y(.) \in Y$ and the control function $u(.) \in U$, $B\colon L^2([0, b], U) \to L^2([0, b], Y_\eta)$ is a bounded linear operator and f is a given nonlinear function.

For some recent works on different types of controllability with Hilfer fractional differential equations, the readers are referred to the works in [4,7, 13,23].

Motivated by the above mentioned works, here we consider the following Hilfer fractional differential equation:

$$
\left.
\begin{aligned}
&D_{0^+}^{\delta,\zeta} y(\tau) = -Ay(\tau) + f(\tau, y(\tau), u(\tau)) + Bu(\tau), \quad \tau \in (0, b],\ b > 0,\\
&I_{0^+}^{(1-\delta)(1-\zeta)} y(0) = y_0,
\end{aligned}
\right\} \tag{1.1}
$$

where $\delta \in (0, 1)$, $\zeta \in [0, 1]$, $-A$ is the infinitesimal generator of an analytic semigroup $\{Q(\tau)\}_{\tau\geq 0}$ on a Banach space Y. The state $y(.)$ takes values in the Banach space Y_η and the control function $u \in L^p([0, b], U)$, where $p\delta > 1$, with U as a Banach space, $B\colon U \to Y_\eta$, $\eta \in (0, 1]$ is a bounded linear operator and

$y_0 \in Y_\eta$. The nonlinear function f satisfies some assumptions which will be specified later.

The approximate controllability of problem (1.1) is established by assuming that a linear system is approximately controllable and the range of the nonlinear function f is contained in the range set of the bounded linear operator B. Here we neither assume any uniform boundedness of the nonlinear function nor any compactness condition on the generated semigroup. Further, the assumptions considered are more general than the assumptions in [9,10,20]. To the best of our knowledge, this type of conditions has not been applied so far for studying the approximate controllability of Hilfer fractional differential systems.

This paper is arranged as follows: in Sect. 2, we recall some definitions and lemmas that are used throughout our work, and also present the definition of mild solution of our considered problem. Section 3 consists of an existence and uniqueness result for the mild solution of (1.1), obtained by using a fixed point theorem. In Sect. 4, we present some sufficient conditions for the approximate controllability of our problem. Section 5 summarizes the findings of this work.

2. Preliminaries

Assume that $(U, \|.\|_U)$ and $(Y, \|.\|_Y)$ are Banach spaces. Without loss of generality, assume that $0 \in \rho(A)$. Then, for any $\eta > 0$, $A^{-\eta}$ is a bounded linear operator defined as

$$A^{-\eta} = \frac{1}{\eta} \int_0^\infty \tau^{\eta-1} Q(\tau) d\tau.$$

Since $A^{-\eta}$ is one-to-one, therefore A^η, for $\eta \geq 0$, is defined as

$$A^\eta = \begin{cases} (A^{-\eta})^{-1}, & \eta > 0, \\ I, & \eta = 0. \end{cases}$$

Furthermore, A^η is a closed linear operator with domain $D(A^\eta) = R(A^{-\eta})$, which is dense in Y. Also, $D(A^\eta)$ is a Banach space with respect to the norm $\|.\|_\eta$ defined by

$$\|y\|_\eta = \|A^\eta y\|_Y \ \ \forall\, y \in D(A^\eta).$$

Denote $Y_\eta = (D(A^\eta), \|.\|_\eta)$. Throughout this work, it is assumed that there exists a constant $M_Q \geq 1$ such that $\|Q(\tau)\|_{B(Y)} \leq M_Q$ for all $\tau \geq 0$, that is, $\{Q(\tau)\}_{\tau\geq 0}$ is uniformly bounded by M_Q, and $\{Q(\tau)\}_{\tau>0}$ is continuous in the uniform operator topology.

Let $J = [0, b]$ and $C(J, Y_\eta)$ denote the Banach space of all continuous functions from J to Y_η. Take $\gamma = \delta+\zeta-\delta\zeta$ so that $1-\gamma = (1-\delta)(1-\zeta) \in [0, 1)$. Define $C_{1-\gamma}(J, Y_\eta) = \{y\colon (0, b] \to Y_\eta | \tau^{1-\gamma} y(\tau) \in C(J, Y_\eta)\}$ which is a Banach space with respect to the norm $\|.\|_{1-\gamma}$ defined by

$$\|y\|_{1-\gamma} = \sup_{\tau\in(0,b]} \tau^{1-\gamma}\|y(\tau)\|_\eta \ \ \forall\, y \in C_{1-\gamma}(J, Y_\eta).$$

Remark: Let $y_1(\tau) = \tau^{\gamma-1}y_2(\tau)$, $\tau \in (0, b]$. Then

$$y_1 \in C_{1-\gamma}(J, Y_\eta) \iff y_2 \in C(J, Y_\eta) \text{ and } \|y_1\|_{1-\gamma} = \|y_2\|_C.$$

Theorem 1. *[18]*

(i) $\eta_1 \geq \eta_2 > 0$ *implies* $D(A^{\eta_1}) \subset D(A^{\eta_2})$,

(ii) if $\eta_1,\ \eta_2 \in \mathbb{R}$, *then* $A^{\eta_1+\eta_2}y = A^{\eta_1}A^{\eta_2}y$, *for every* $y \in D(A^\eta)$ *where* $\eta = \max\{\eta_1, \eta_2, \eta_1 + \eta_2\}$.

Lemma 1. [18] *There exists a constant* $M_\eta > 0$ *such that* $\|A^{-\eta}\| \leq M_\eta$, *for* $\eta \in [0, 1]$.

Theorem 2. [18]

(i) $Q(\tau)\colon Y \to D(A^\eta)$ *for every* $\tau > 0$ *and* $\eta \geq 0$,

(ii) for every $y \in D(A^\eta)$, $Q(\tau)A^\eta y = A^\eta Q(\tau)y$,

(ii) for every $\tau > 0$, $A^\eta Q(\tau)$ *is bounded and there exists a constant* $C_\eta > 0$ *such that* $\|A^\eta Q(\tau)\| \leq \frac{C_\eta}{\tau^\eta}$.

Remark. [24] Let $Q_\eta(\tau)$ be the restriction of $Q(\tau)$ to Y_η. Then, $\{Q_\eta(\tau)\}_{\tau\geq 0}$ is a family of bounded linear operators on Y_η and satisfies $\|Q_\eta(\tau)\| \leq \|Q(\tau)\|$ for all $\tau \geq 0$. Moreover, $\{Q_\eta(\tau)\}_{\tau\geq 0}$ forms a C_0-semigroup on Y_η.

Definition 1. [3] The left-sided Riemann-Liouville fractional integral of order $\delta > 0$ of a function f with lower limit 0 is defined as

$$I^\delta_{0+}f(\tau) := \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau - \phi)^{\delta-1}f(\phi)d\phi, \ \ \tau > 0.$$

Definition 2. [3] The left-sided Riemann-Liouville derivative of order $\delta > 0$ of a function f with lower limit 0 is defined as

$${}^{RL}D^\delta_{0+}f = D^n I^{n-\delta}_{0+}f,$$

with n denoting the greatest integer less than or equal to δ and D^n denoting the n-th order differential operator. For $\delta = 0$, ${}^{RL}D^\delta_{0+} = I$, the identity operator.

Definition 3. [26] The left-sided Caputo derivative of order $\delta > 0$ of a function f with lower limit 0 is defined as

$$\begin{aligned}{}^{C}D^\delta_{0+}f(\tau) =& \frac{1}{\Gamma(n-\delta)}\int_0^\tau (\tau - \phi)^{n-\delta-1}f^{(n)}(\phi)d\phi, \ \ \tau > 0\\ =& I^{n-\delta}_{0+}f^n(\tau).\end{aligned}$$

For $\delta = 0$, ${}^{C}D^\delta_{0+} = I$.

Definition 4. [26] The left-sided Hilfer derivative of order $\delta \in (0,1)$ and type $\zeta \in [0,1]$ is defined as

$$D^{\delta,\zeta}_{0^+} = I^{\zeta(1-\delta)}_{0^+} D I^{(1-\zeta)(1-\delta)}_{0^+},$$

where D denotes the differential operator.

Also, we have $D^{\delta,0}_{0^+} = DI^{1-\delta}_{0^+} = {}^{RL}D^{\delta}_{0^+}$, and $D^{\delta,1}_{0^+} = I^{1-\delta}_{0^+}D = {}^{C}D^{\delta}_{0^+}$.

Let $y(t; y_0, u)$ denote the state value of (1.1) at time t corresponding to the initial value y_0 and control $u(.)$. Then, we have the following definition of mild solution:

Definition 5. [6,26] A function $y(.; y_0, u) \in C_{1-\gamma}(J, Y_\eta)$ is said to be a mild solution of (1.1), if for any u in $L^p([0, b], U)$, the following integral equation is satisfied:

$$\begin{aligned} y(\tau; y_0, u) &= T_{\zeta,\delta}(\tau)y_0 + \int_0^\tau S_\delta(\tau-\phi)\big[f(\phi, y(\phi), u(\phi)) + Bu(\phi)\big]d\phi \\ &= T_{\zeta,\delta}(\tau)y_0 + \int_0^\tau (\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)f(\phi, y(\phi), u(\phi))d\phi \\ &\quad + \int_0^\tau (\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)Bu(\phi)d\phi \end{aligned}$$

for $\tau \in (0, b]$, where

$$T_{\zeta,\delta}(\tau) = I^{\zeta(1-\delta)}_{0^+}S_\delta(\tau),\ S_\delta(\tau) = \tau^{\delta-1}R_\delta(\tau),\ R_\delta(\tau) = \int_0^\infty \delta\theta M_\delta(\theta)Q(\tau^\delta\theta)d\theta,$$

$$M_\delta(\theta) = \frac{1}{\delta}\theta^{-1-\frac{1}{\delta}}\psi_\delta(\theta^{-\frac{1}{\delta}}),\ \psi_\delta(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{(-\delta n-1)}\frac{\Gamma(\delta n+1)}{n!}\sin(n\pi\delta),$$

for $\theta \in (0,\infty)$. Here, $M_\delta(\theta)$ is a probability density function on $(0,\infty)$ satisfying

$$M_\delta(\theta) \geq 0,\ \int_0^\infty M_\delta(\theta)d\theta = 1,\ \int_0^\infty \theta M_\delta(\theta)d\theta = \frac{1}{\Gamma(1+\delta)}.$$

Next we have the following properties of the solution operators $R_\delta(\tau)$, $S_\delta(\tau)$ and $T_{\zeta,\delta}(\tau)$ [6]:

(P1) $R_\delta(\tau)$ is continuous in the uniform operator topology for $\tau > 0$.

(P2) for any fixed $\tau > 0$, $R_\delta(\tau)$, $S_\delta(\tau)$ and $T_{\zeta,\delta}(\tau)$ are linear operators on Y, and

$$\|R_\delta(\tau)y\|_Y \leq \frac{M_Q}{\Gamma(\delta)}\|y\|_Y,$$

$$\|S_\delta(\tau)y\|_Y \leq \frac{M_Q\tau^{\delta-1}}{\Gamma(\delta)}\|y\|_Y,$$

$$\|T_{\zeta,\delta}(\tau)y\|_Y \leq \frac{M_Q\tau^{\gamma-1}}{\Gamma(\gamma)}\|y\|_Y$$

hold for any $y \in Y$.

(P3) $\{S_\delta(\tau)\}_{\tau>0}$ and $\{T_{\zeta,\delta}(\tau)\}_{\tau>0}$ are strongly continuous.

Before going to the next step, let us first recall some of the remaining properties:

(P4) for any fixed $\tau > 0$, and any $y \in Y_\eta$,

$$\|R_\delta(\tau)y\|_\eta \leq \frac{M_Q}{\Gamma(\delta)}\|y\|_\eta,$$

$$\|S_\delta(\tau)y\|_\eta \leq \frac{M_Q\tau^{\delta-1}}{\Gamma(\delta)}\|y\|_\eta,$$

$$\|T_{\zeta,\delta}(\tau)y\|_\eta \leq \frac{M_Q\tau^{\gamma-1}}{\Gamma(\gamma)}\|y\|_\eta.$$

(P5) For each $y \in Y_\eta$ and $\tau > 0$,

$$A^\eta R_\delta(\tau)y = R_\delta(\tau)A^\eta y, \quad A^\eta S_\delta(\tau)y = S_\delta(\tau)A^\eta y, \quad A^\eta T_{\zeta,\delta}(\tau)y = T_{\zeta,\delta}(\tau)A^\eta y.$$

Theorem 3. [15] *Let Y be a Banach space and $F\colon Y \to Y$ be a map such that $F^{(i)}$ $(= \underbrace{F \circ F \circ \ldots \circ F}_{i\ times})$ is a contraction map for some $i \in \mathbb{N}$. Then, F has a unique fixed point on Y.*

Definition 6. [14] The one-parameter Mittag-Leffler function is defined by

$$E_c(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(cn+1)}, \quad z \in \mathbb{C}, \ \mathrm{Re}(c) > 0,$$

where Re denotes the real part.

This function is a generalization of the exponential function. An extension of the above function is the following two-parameter Mittag-Leffler function:

$$E_{c,d}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(cn+d)}, \quad z, d \in \mathbb{C}, \ \mathrm{Re}(c) > 0.$$

Lemma 2. [16,27] *Assume that $\alpha_1\colon [0,b] \to [0,\infty)$ is locally integrable and $\alpha_2\colon [0,b] \to [0,\infty)$ is a nondecreasing continuous function such that $\alpha_2(\tau) \leq \mathcal{C}$ (a constant). Suppose $\alpha_3\colon [0,b] \to [0,\infty)$ is locally integrable and satisfies the inequality*

$$\alpha_3(\tau) \leq \alpha_1(\tau) + \alpha_2(\tau)\int_0^\tau (\tau-\phi)^{r-1}\alpha_3(\phi)d\phi, \quad \tau \in [0,b], \ r > 0.$$

Then

$$\alpha_3(\tau) \leq \alpha_1(\tau) + \int_0^\tau \left[\sum_{i=1}^{\infty} \frac{(\alpha_2(\tau)\Gamma(r))^i}{\Gamma(ir)}(\tau-\phi)^{ir-1}\alpha_1(\phi)\right]d\phi, \quad \tau \in [0,b].$$

In addition, if α_1 is nondecreasing, then $\alpha_3(\tau) \leq \alpha_1(\tau)E_r(\alpha_2(\tau)\Gamma(r)\tau^r)$ for $\tau \in [0,b]$.

3. Existence and Uniqueness of Mild Solution

Take $\zeta \neq 0$, that is, $\zeta \in (0, 1]$, then we have the following limits [6]:

$$\lim_{\tau \to 0^+} \tau^{1-\gamma} T_{\zeta,\delta}(\tau) y_0 = \frac{y_0}{\Gamma(\gamma)},$$
$$\lim_{\tau \to 0^+} \tau^{1-\gamma} \int_0^\tau (\tau - \phi)^{\delta-1} R_\delta(\tau - \phi) f(\phi, y(\phi), u(\phi)) d\phi = 0,$$
$$\lim_{\tau \to 0^+} \tau^{1-\gamma} \int_0^\tau (\tau - \phi)^{\delta-1} R_\delta(\tau - \phi) B u(\phi) d\phi = 0.$$

Also, let $a = \frac{(\delta-1)p}{p-1}$.

For the existence and uniqueness result, we use the following assumptions:
(Hf) there exists a constant $\xi \in [\eta, 1]$ such that $f\colon [0, b] \times Y_\eta \times U \to Y_\xi$ satisfies the following:

(i) there exists a constant $L > 0$ such that
$$\|f(\tau, y_1, u_1) - f(\tau, y_2, u_2)\|_\xi \leq L\big[\|y_1 - y_2\|_\eta + \|u_1 - u_2\|_U\big]$$
for all $y_i \in Y_\eta$, $u_i \in U$; $i = 1, 2$ and $\tau \in [0, b]$.

(ii) there exist a function $g \in L^p([0, b], [0, \infty))$ and a constant $c > 0$ such that
$$\|f(\tau, y, u)\|_\xi \leq g(\tau) + c\big(\tau^{1-\gamma}\|y\|_\eta + \|u\|_U\big)$$
for all $y \in Y_\eta$, $u \in U$ and $\tau \in [0, b]$.

Theorem 4. *If the above assumptions are satisfied, then for each $u \in L^p([0, b], U)$, problem (1.1) has a unique mild solution on $C_{1-\gamma}(J, Y_\eta)$.*

Proof. Define a map Υ on $C_{1-\gamma}(J, Y_\eta)$ by
$$(\Upsilon y)(\tau) = T_{\zeta,\delta}(\tau) y_0 + \int_0^\tau (\tau - \phi)^{\delta-1} R_\delta(\tau - \phi)[f(\phi, y(\phi), u(\phi)) + Bu(\phi)] d\phi.$$
□

The proof is split into several parts as follows.

Step 1: To show that Υ is well-defined on $C_{1-\gamma}(J, Y_\eta)$:

Using (P4), Theorem 1, Lemma 1, (Hf)(ii) and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\tau \|(\tau - \phi)^{\delta-1} R_\delta(\tau - \phi) f(\phi, y(\phi), u(\phi))\|_\eta d\phi \\
&\leq \frac{M_Q}{\Gamma(\delta)} \int_0^\tau (\tau - \phi)^{\delta-1} \|f(\phi, y(\phi), u(\phi))\|_\eta d\phi \\
&\leq \frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)} \int_0^\tau (\tau - \phi)^{\delta-1} \|f(\phi, y(\phi), u(\phi))\|_\xi d\phi \\
&\leq \frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)} \int_0^\tau (\tau - \phi)^{\delta-1} \big[g(\phi) + c\big(\phi^{1-\gamma}\|y(\phi)\|_\eta + \|u(\phi)\|_U\big)\big] d\phi
\end{aligned}$$

$$\leq \frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)}\left[\left(\|g\|_{L^p}+c\|u\|_{L^p}\right)\frac{b^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}+\frac{cb^\delta}{\delta}\|y\|_{1-\gamma}\right].$$

Similarly, using (P4) and Hölder's inequality,

$$\begin{aligned}\int_0^\tau \|(\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)Bu(\phi)\|_\eta d\phi \leq &\frac{M_Q}{\Gamma(\delta)}\int_0^\tau(\tau-\phi)^{\delta-1}\|Bu(\phi)\|_\eta d\phi\\ \leq &\frac{M_Q}{\Gamma(\delta)}\|Bu\|_{L^p}\frac{b^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}.\end{aligned}$$

Therefore, $(\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)f(\phi,y(\phi),u(\phi))$ and $(\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)Bu(\phi)$ are Bochner integrable w.r.t. $\phi\in[0,\tau]$ for all $\tau\in(0,b]$. Hence, $(\Upsilon y)(.)$ is well-defined on $(0,b]$ for any $y\in C_{1-\gamma}(J,Y_\eta)$.

Step 2: To show that $\Upsilon y\in C_{1-\gamma}(J,Y_\eta)$ for any $y\in C_{1-\gamma}(J,Y_\eta)$:

Let $y\in C_{1-\gamma}(J,Y_\eta)$. Define $y\colon[0,b]\to Y_\eta$ by

$$\begin{aligned}y(\tau)=&\begin{cases}\lim_{\tau\to 0}\tau^{1-\gamma}(\Upsilon y)(\tau), & \tau=0,\\ \tau^{1-\gamma}(\Upsilon y)(\tau), & \tau\in(0,b],\end{cases}\\ =&\begin{cases}\frac{y_0}{\Gamma(\gamma)}, & \tau=0,\\ \tau^{1-\gamma}(\Upsilon y)(\tau), & \tau\in(0,b].\end{cases}\end{aligned}$$

Then, it can be easily seen that, for $0=\tau_1<\tau_2\leq b$,

$$\|y(\tau_2)-y(\tau_1)\|_\eta\longrightarrow 0 \text{ as } \tau_2\to\tau_1.$$

Next, for $0<\tau_1<\tau_2\leq b$, we have

$$\begin{aligned}&\|y(\tau_2)-y(\tau_1)\|_\eta\\ &\quad\leq\|\tau_2^{1-\gamma}T_{\zeta,\delta}(\tau_2)y_0-\tau_1^{1-\gamma}T_{\zeta,\delta}(\tau_1)y_0\|_\eta+\bigg\|\tau_2^{1-\gamma}\int_0^{\tau_2}(\tau_2-\phi)^{\delta-1}R_\delta(\tau_2-\phi)\\ &\times f(\phi,y(\phi),u(\phi))d\phi-\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}R_\delta(\tau_1-\phi)f(\phi,y(\phi),u(\phi))d\phi\bigg\|_\eta\\ &\quad+\bigg\|\tau_2^{1-\gamma}\int_0^{\tau_2}(\tau_2-\phi)^{\delta-1}R_\delta(\tau_2-\phi)Bu(\phi)d\phi-\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}R_\delta(\tau_1-\phi)\\ &\quad\times Bu(\phi)d\phi\bigg\|_\eta.\end{aligned}$$

First term:

$$\begin{aligned}
&\|\tau_2^{1-\gamma} T_{\zeta,\delta}(\tau_2)y_0 - \tau_1^{1-\gamma} T_{\zeta,\delta}(\tau_1)y_0\|_\eta \\
&\quad= \left\| \frac{1}{\Gamma(\zeta(1-\delta))}\left[\tau_2^{1-\gamma}\int_0^{\tau_2}(\tau_2-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}R_\delta(\phi)y_0 d\phi - \tau_1^{1-\gamma}\right.\right. \\
&\qquad\left.\left. \times \int_0^{\tau_1}(\tau_1-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}R_\delta(\phi)y_0 d\phi\right]\right\|_\eta \\
&\quad\leq \left\|\frac{1}{\Gamma(\zeta(1-\delta))}\int_{\tau_1}^{\tau_2}\tau_2^{1-\gamma}(\tau_2-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}R_\delta(\phi)y_0 d\phi\right\|_\eta + \frac{1}{\Gamma(\zeta(1-\delta))} \\
&\qquad \times \left\|\int_0^{\tau_1}\tau_2^{1-\gamma}\{(\tau_2-\phi)^{\zeta(1-\delta)-1} - (\tau_1-\phi)^{\zeta(1-\delta)-1}\}\phi^{\delta-1}R_\delta(\phi)y_0 d\phi\right\|_\eta \\
&\qquad + \left\|\frac{\tau_2^{1-\gamma}-\tau_1^{1-\gamma}}{\Gamma(\zeta(1-\delta))}\left[\int_0^{\tau_1}(\tau_1-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}R_\delta(\phi)y_0 d\phi\right]\right\|_\eta \\
&\quad\leq I_{11} + I_{12} + I_{13},
\end{aligned}$$

where

$$\begin{aligned}
I_{11} &= \frac{M_Q\|y_0\|_\eta \tau_2^{1-\gamma}}{\Gamma(\delta)\Gamma(\zeta(1-\delta))}\int_{\tau_1}^{\tau_2}(\tau_2-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}d\phi, \\
I_{12} &= \frac{M_Q\tau_2^{1-\gamma}\|y_0\|_\eta}{\Gamma(\zeta(1-\delta))}\int_0^{\tau_1}[(\tau_1-\phi)^{\zeta(1-\delta)-1} - (\tau_2-\phi)^{\zeta(1-\delta)-1}]\phi^{\delta-1}d\phi, \\
I_{13} &= \frac{\tau_2^{1-\gamma}-\tau_1^{1-\gamma}}{\Gamma(\zeta(1-\delta))}\int_0^{\tau_1}(\tau_1-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}\|R_\delta(\phi)y_0\|_\eta d\phi.
\end{aligned}$$

By absolute continuity of Lebesgue integral, $I_{11} \to 0$ as $\tau_2 \to \tau_1$. For I_{12}, we have

$$[(\tau_1-\phi)^{\zeta(1-\delta)-1} - (\tau_2-\phi)^{\zeta(1-\delta)-1}]\phi^{\delta-1} \leq (\tau_1-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}$$

for a.e. $\phi \in [0, \tau_1]$. Therefore, by vector-valued dominated convergence theorem, $I_{12} \to 0$ as $\tau_2 \to \tau_1$. Next, for I_{13}, we have

$$\begin{aligned}
I_{13} &\leq \frac{M_Q(\tau_2^{1-\gamma}-\tau_1^{1-\gamma})\|y_0\|_\eta}{\Gamma(\delta)\Gamma(\zeta(1-\delta))}\int_0^{\tau_1}(\tau_1-\phi)^{\zeta(1-\delta)-1}\phi^{\delta-1}d\phi \\
&= \frac{M_Q\tau_1^{\gamma-1}(\tau_2^{1-\gamma}-\tau_1^{1-\gamma})\|y_0\|_\eta}{\Gamma(\delta)\Gamma(\zeta(1-\delta))} \longrightarrow 0, \;\; \text{as } \tau_2 \to \tau_1.
\end{aligned}$$

Second term:

$$\begin{aligned}
&\Big\|\tau_2^{1-\gamma}\int_0^{\tau_2}(\tau_2-\phi)^{\delta-1}R_\delta(\tau_2-\phi)f(\phi,y(\phi),u(\phi))d\phi \\
&\quad -\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}R_\delta(\tau_1-\phi)f(\phi,y(\phi),u(\phi))d\phi\Big\|_\eta \\
&\quad\le \tau_2^{1-\gamma}\int_{\tau_1}^{\tau_2}(\tau_2-\phi)^{\delta-1}\big\|R_\delta(\tau_2-\phi)f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi+\int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1} \\
&\quad -\tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|\big\|R_\delta(\tau_2-\phi)f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi+\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1} \\
&\quad \times\big\|[R_\delta(\tau_2-\phi)-R_\delta(\tau_1-\phi)]f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi \\
&\quad = I_{21}+I_{22}+I_{23},
\end{aligned}$$

where

$$\begin{aligned}
I_{21}&=\tau_2^{1-\gamma}\int_{\tau_1}^{\tau_2}(\tau_2-\phi)^{\delta-1}\big\|R_\delta(\tau_2-\phi)f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi,\\
I_{22}&=\int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1}-\tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|\big\|R_\delta(\tau_2-\phi)f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi,\\
I_{23}&=\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}\big\|[R_\delta(\tau_2-\phi)-R_\delta(\tau_1-\phi)]f(\phi,y(\phi),u(\phi))\big\|_\eta d\phi.
\end{aligned}$$

Now,

$$\begin{aligned}
I_{21}&\le\frac{M_Q M_{\xi-\eta}\tau_2^{1-\gamma}}{\Gamma(\delta)}\int_{\tau_1}^{\tau_2}(\tau_2-\phi)^{\delta-1}\big\|f(\phi,y(\phi),u(\phi))\big\|_\xi d\phi\\
&\le\frac{M_Q M_{\xi-\eta}b^{1-\gamma}}{\Gamma(\delta)}\left[\big(\|g\|_{L^p}+c\|u\|_{L^p}\big)\frac{(\tau_2-\tau_1)^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}+\frac{c(\tau_2-\tau_1)^\delta}{\delta}\|y\|_{1-\gamma}\right]\\
&\quad\longrightarrow 0,\ \text{as } \tau_2\to\tau_1,\\
I_{22}&\le\frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)}\bigg[\int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1}-\tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|g(\phi)d\phi+c\|y\|_{1-\gamma}\\
&\quad\times\int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1}-\tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|d\phi+c\int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1}\\
&\quad-\tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|\|u(\phi)\|_U d\phi\bigg]
\end{aligned}$$

which converges to 0 as $\tau_2\to\tau_1$ due to Lebesgue's dominated convergence theorem.

Next, for $\epsilon > 0$ small enough, we have

$$I_{23} \leq M_{\xi-\eta}\Bigg[\big(\|g\|_{L^p} + c\|u\|_{L^p}\big)\frac{\tau_1^{1-\gamma+\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}} + c\|y\|_{1-\gamma}\frac{\tau_1{}^{1-\gamma+\delta}}{\delta}\Bigg]$$
$$\times \sup_{\phi\in[0,\tau_1-\epsilon]}\big\|R_\delta(\tau_2-\phi) - R_\delta(\tau_1-\phi)\big\|_{B(Y)} + \frac{2M_Q M_{\xi-\eta}b^{1-\gamma}}{\Gamma(\delta)}$$
$$\times \Bigg[\big(\|g\|_{L^p} + c\|u\|_{L^p}\big)\frac{\epsilon^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}} + c\|y\|_{1-\gamma}\frac{\epsilon^\delta}{\delta}\Bigg],$$

where the RHS converges to zero, by using (P1), as $\tau_2 \to \tau_1$ and $\epsilon \to 0$.
Third term:

$$\Big\|\tau_2^{1-\gamma}\int_0^{\tau_2}(\tau_2-\phi)^{\delta-1}R_\delta(\tau_2-\phi)Bu(\phi)d\phi$$
$$-\tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}R_\delta(\tau_1-\phi)Bu(\phi)d\phi\Big\|_\eta \leq I_{31} + I_{32} + I_{33},$$

where

$$I_{31} = \tau_2^{1-\gamma}\int_{\tau_1}^{\tau_2}(\tau_2-\phi)^{\delta-1}\|R_\delta(\tau_2-\phi)Bu(\phi)\|_\eta d\phi,$$
$$I_{32} = \int_0^{\tau_1}\big|\tau_2^{1-\gamma}(\tau_2-\phi)^{\delta-1} - \tau_1^{1-\gamma}(\tau_1-\phi)^{\delta-1}\big|\|R_\delta(\tau_2-\phi)Bu(\phi)\|_\eta d\phi,$$
$$I_{33} = \tau_1^{1-\gamma}\int_0^{\tau_1}(\tau_1-\phi)^{\delta-1}\big\|[R_\delta(\tau_2-\phi) - R_\delta(\tau_1-\phi)]Bu(\phi)\big\|_\eta d\phi.$$

Now,

$$I_{31} \leq \frac{M_Q\|Bu\|_{L^p}b^{1-\gamma}}{(a+1)^{\frac{(p-1)}{p}}\Gamma(\delta)}(\tau_2-\tau_1)^{\frac{(\delta p-1)}{p}} \longrightarrow 0, \text{ as } \tau_2 \to \tau_1.$$

By Lebesgue's dominated convergence theorem, $I_{32} \to 0$ as $\tau_2 \to \tau_1$, and applying a similar technique as for I_{23}, we get $I_{33} \to 0$ as $\tau_2 \to \tau_1$.

Thus, we have for $0 < \tau_1 < \tau_2 \leq b$,

$$\|y(\tau_2) - y(\tau_1)\|_\eta \longrightarrow 0, \text{ as } \tau_2 \to \tau_1.$$

Therefore, $y \in C(J, Y_\eta)$ and hence $\Upsilon y \in C_{1-\gamma}(J, Y_\eta)$.

Step 3: To show that $\Upsilon^{(i)}$ is a contraction for some $i \in \mathbb{N}$:

We proceed by induction on i. Let x, $y \in C_{1-\gamma}(J, Y_\eta)$. Then, for any $\tau \in (0, b]$, we claim that

$$\tau^{1-\gamma}\|(\Upsilon^{(i)}x)(\tau) - (\Upsilon^{(i)}y)(\tau)\|_\eta \leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta}\tau^\delta)^i}{\Gamma(i\delta+\gamma)}\|x-y\|_{1-\gamma} \text{ for all } i \in \mathbb{N}. \tag{3.1}$$

For $i = 1$, using (P4), Lemma 1 and (Hf)(i), we have

$$\begin{aligned}\tau^{1-\gamma}\|(\Upsilon x)(\tau) - (\Upsilon y)(\tau)\|_\eta &\leq \frac{M_Q M_{\xi-\eta}\tau^{1-\gamma}}{\Gamma(\delta)}\int_0^\tau (\tau-\phi)^{\delta-1}\|f(\phi, x(\phi), u(\phi)) \\ &\quad - f(\phi, y(\phi), u(\phi))\|_\xi d\phi \\ &\leq \frac{LM_Q M_{\xi-\eta}\tau^{1-\gamma}}{\Gamma(\delta)}\int_0^\tau (\tau-\phi)^{\delta-1}\|x(\phi)-y(\phi)\|_\eta d\phi \\ &\leq \frac{LM_Q M_{\xi-\eta}\tau^{1-\gamma}}{\Gamma(\delta)}\|x-y\|_{1-\gamma}\int_0^\tau (\tau-\phi)^{\delta-1}\phi^{\gamma-1} d\phi \\ &= \Gamma(\gamma)\frac{LM_Q M_{\xi-\eta}\tau^\delta}{\Gamma(\delta+\gamma)}\|x-y\|_{1-\gamma}.\end{aligned}$$

Thus, Eq. (3.1) holds for $i = 1$.
Induction hypothesis: Assume that (3.1) holds for $i = k$, i.e.,

$$\tau^{1-\gamma}\|(\Upsilon^{(k)}x)(\tau) - (\Upsilon^{(k)}y)(\tau)\|_\eta \leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta}\tau^\delta)^k}{\Gamma(k\delta+\gamma)}\|x-y\|_{1-\gamma}.$$

Then,

$$\begin{aligned}&\tau^{1-\gamma}\|(\Upsilon^{(k+1)}x)(\tau) - (\Upsilon^{(k+1)}y)(\tau)\|_\eta \\ &\quad\leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta})^{k+1}\tau^{1-\gamma}}{\Gamma(k\delta+\gamma)\Gamma(\delta)}\|x-y\|_{1-\gamma}\int_0^\tau (\tau-\phi)^{\delta-1}\phi^{\gamma+k\delta-1}d\phi \\ &\quad\leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta}\tau^\delta)^{k+1}}{\Gamma((k+1)\delta+\gamma)}\|x-y\|_{1-\gamma}.\end{aligned}$$

Thus, by principle of mathematical induction, (3.1) holds for all $i \in \mathbb{N}$.
Now, for $\tau \in (0, b]$, we have

$$\tau^{1-\gamma}\|(\Upsilon^{(i)}x)(\tau) - (\Upsilon^{(i)}y)(\tau)\|_\eta \leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta}\tau^\delta)^i}{\Gamma(i\delta+\gamma)}\|x-y\|_{1-\gamma}$$

which gives

$$\|\Upsilon^{(i)}x - \Upsilon^{(i)}y\|_{1-\gamma} \leq \Gamma(\gamma)\frac{(LM_Q M_{\xi-\eta}b^\delta)^i}{\Gamma(i\delta+\gamma)}\|x-y\|_{1-\gamma}.$$

Since the series $E_{\delta,\gamma}(LM_Q M_{\xi-\eta}b^\delta) = \sum_{i=0}^{\infty}\frac{(LM_Q M_{\xi-\eta}b^\delta)^i}{\Gamma(i\delta+\gamma)}$ converges, therefore we can get

$$\frac{(LM_Q M_{\xi-\eta}b^\delta)^i}{\Gamma(i\delta+\gamma)} < \frac{1}{\Gamma(\gamma)} \quad \text{for } i \text{ sufficiently large.}$$

Therefore, $\Upsilon^{(i)}$ is a contraction on $C_{1-\gamma}(J, Y_\eta)$ and thus Υ has a unique fixed point on $C_{1-\gamma}(J, Y_\eta)$.

In other words, problem (1.1) has a unique solution on $C_{1-\gamma}(J, Y_\eta)$.

4. Approximate Controllability

In this section, we establish the approximate controllability of Eq. (1.1).

We know that $y(b; y_0, u)$ denotes the state value of (1.1) at terminal time b corresponding to the initial value y_0 and control $u(.)$. Let $R_b(f) = \{y(b; y_0, u) | u \in L^p([0, b], U)\}$ denote the reachable set – the set of all points to which the initial state y_0 can be steered in time b under the influence of the control u.

Definition 7. (1.1) is said to be approximately controllable on $[0, b]$ if given an arbitrary $\epsilon > 0$, it is possible to steer from y_0 to a point within a distance ϵ from all points in the state space Y_η at time b.

Thus, in terms of the reachable set, (1.1) is approximately controllable on $[0, b]$ if and only if $\overline{R_b(f)} = Y_\eta$.

Now, consider the following linear system:

$$\left.\begin{aligned} &D_{0+}^{\delta,\zeta} y(\tau) = -Ay(\tau) + Bv(\tau), \quad \tau \in (0, b], \\ &I_{0+}^{(1-\delta)(1-\zeta)} y(0) = y_0. \end{aligned}\right\} \tag{4.1}$$

Then in accordance with the above notation, the reachable set of (4.1) is denoted by $R_b(0)$.

Fix $y \in C_{1-\gamma}([0, b], Y_\eta)$ and $h \in L^p([0, b], U)$. Define a map K by

$$K(\tau) = f(\tau, y(\tau), h(\tau)).$$

Then K belongs to $L^p([0, b], Y_\eta)$.

To prove the approximate controllability of (1.1), we use the generalized Grönwall inequality (Lemma 2), for which we need to modify our assumption (i) in (Hf) as

(iii) there exists a constant $N > 0$ such that

$$\|f(\tau, y_1, u_1) - f(\tau, y_2, u_2)\|_\xi \leq N\tau^{1-\gamma}\left[\|y_1 - y_2\|_\eta + \|u_1 - u_2\|_U\right],$$

for all $y_i \in Y_\eta$, $u_i \in U$; $i = 1, 2$ and $\tau \in [0, b]$.

Also, we consider the following assumptions:

(HfB) range$(f) \subset$ range(B).

(HB) there exists a constant $e > 0$ such that $\|Bu\|_\eta \geq e\|u\|_U$ for all $u \in U$.

Theorem 5. *Assume that hypotheses (HfB), (HB) and (Hf) (with (i) replaced by (iii)) hold. Then, the approximate controllability of linear system (4.1) and the inequality*

$$\max\{cM_{\xi-\eta}, M_{\xi-\eta}Nb^{1-\gamma}\} < e$$

imply that (1.1) is approximately controllable.

Proof. Let w be the mild solution of the linear system (4.1) corresponding to a control v. First, we show that for $w \in C_{1-\gamma}(J, Y_\eta)$ and $v \in L^p([0, b], U)$, there exists a control function $u \in L^p([0, b], U)$ such that it satisfies $Bu(\tau) = Bv(\tau) - f(\tau, w(\tau), u(\tau))$, $\tau \in (0, b]$. □

Now, define a new function $\Pi\colon \text{range}(B) \subset Y_\eta \to U$ by

$$\Pi r = u \;\text{ whenever }\; Bu = r.$$

Then, using (HB), it can be shown that Π is well-defined. The function Π forms a bounded linear map with $\|\Pi\| \le \frac{1}{e}$. Also, $\Pi B = Id_U$ and $B\Pi = Id_{range(B)}$.

Next, we begin by showing that for each $\tau \in (0, b]$, there exists $u(t) \in U$ such that $u(\tau) = v(\tau) - \Pi f(\tau, w(\tau), u(\tau))$, for all $\tau \in (0, b]$.

Let $h_0(\tau) = v(\tau)$, $\tau \in [0, b]$ and for each $n \in \mathbb{N}$, define

$$h_n(\tau) = \begin{cases} v(\tau) - \Pi f(\tau, w(\tau), h_{n-1}(\tau)), & \tau \in (0, b], \\ v(0), & \tau = 0. \end{cases}$$

Then by fixing $\tau \in (0, b]$,

$$\begin{aligned} \|h_{n+1}(\tau) - h_n(\tau)\|_U &= \|\Pi f(\tau, w(\tau), h_n(\tau)) - \Pi f(\tau, w(\tau), h_{n-1}(\tau))\|_U \\ &\le \frac{1}{e} M_{\xi-\eta} \|f(\tau, w(\tau), h_n(\tau)) - f(\tau, w(\tau), h_{n-1}(\tau))\|_\xi \\ &\le \frac{1}{e} M_{\xi-\eta} N \tau^{1-\gamma} \|h_n(\tau) - h_{n-1}(\tau)\|_U \\ &\le \left(\frac{1}{e} M_{\xi-\eta} N b^{1-\gamma}\right)^n \|h_1(\tau) - h_0(\tau)\|_U, \end{aligned}$$

and for $m > n$ $(m, n \in \mathbb{N})$,

$$\begin{aligned} &\|h_m(\tau) - h_n(\tau)\|_U \\ \le& \|h_m(\tau) - h_{m-1}(\tau)\|_U + \|h_{m-1}(\tau) - h_{m-2}(\tau)\|_U + \cdots + \|h_{n+1}(\tau) - h_n(\tau)\|_U \\ \le& \left(\frac{M_{\xi-\eta} N b^{1-\gamma}}{e}\right)^n \frac{1 - \left(\frac{M_{\xi-\eta} N b^{1-\gamma}}{e}\right)^{m-n}}{1 - \frac{M_{\xi-\eta} N b^{1-\gamma}}{e}} \|h_1(\tau) - h_0(\tau)\|_U \\ \le& \left(\frac{M_{\xi-\eta} N b^{1-\gamma}}{e}\right)^n \frac{1}{1 - \frac{M_{\xi-\eta} N b^{1-\gamma}}{e}} \|h_1(\tau) - h_0(\tau)\|_U \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $(h_n(\tau)) \subset U$ is a Cauchy sequence, and U being complete, we have $\lim_{n\to\infty} h_n(\tau) \in U$. Since, this argument holds for each $\tau \in (0, b]$, we define a function $u\colon [0, b] \to U$ by

$$u(\tau) = \begin{cases} \lim\limits_{n\to\infty} h_n(\tau), & \tau \in (0, b], \\ v(0), & \tau = 0. \end{cases}$$

Again, for each fixed $\tau \in (0, b]$,

$$\begin{aligned}
&\|v(\tau) - h_{n+1}(\tau) - \Pi f(\tau, w(\tau), u(\tau))\|_U \\
&\quad \leq \frac{M_{\xi-\eta}}{e} \|f(\tau, w(\tau), h_n(\tau)) - f(\tau, w(\tau), u(\tau))\|_\xi \\
&\quad \leq \frac{M_{\xi-\eta} N b^{1-\gamma}}{e} \|h_n(\tau) - u(\tau)\|_U \\
&\quad\quad \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $u(\tau) = v(\tau) - \Pi f(\tau, w(\tau), u(\tau))$, for each $\tau \in (0, b]$.

Now, it remains to show that the function u belongs to $L^p([0, b], U)$. Observe that, for each $n \in \mathbb{N} \cup \{0\}$, $h_n \in L^p([0, b], U)$, because by definition $h_0 \in L^p([0, b], U)$, and if we define $K_n(\tau) = f(\tau, w(\tau), h_{n-1}(\tau))$, then $K_n \in L^p([0, b], Y_\eta)$ and therefore $\Pi K_n \in L^p([0, b], U)$.
Also,

$$\begin{aligned}
&\|h_n(\tau)\|_U \\
&\quad \leq \|v(\tau)\|_U + \frac{M_{\xi-\eta}}{e} \|f(\tau, w(\tau), h_{n-1}(\tau))\|_\xi \\
&\quad \leq \|v(\tau)\|_U + \frac{M_{\xi-\eta}}{e} g(\tau) + \frac{M_{\xi-\eta}}{e} c\tau^{1-\gamma} \|w(\tau)\|_\eta + \frac{M_{\xi-\eta}}{e} c \|h_{n-1}(\tau)\|_U \\
&\quad \leq \left(1 + \frac{cM_{\xi-\eta}}{e}\right) \|v(\tau)\|_U + \left(\frac{M_{\xi-\eta}}{e} + c\frac{M_{\xi-\eta}^2}{e^2}\right) g(\tau) + \left(\frac{M_{\xi-\eta}}{e} c\tau^{1-\gamma} + \frac{M_{\xi-\eta}^2}{e^2} c^2 \tau^{1-\gamma}\right) \\
&\quad\quad \times \|w(\tau)\|_\eta + \frac{M_{\xi-\eta}^2}{e^2} c^2 \|h_{n-2}(\tau)\|_U \\
&\quad \leq \left[1 + \frac{cM_{\xi-\eta}}{e} + \cdots + \frac{c^n M_{\xi-\eta}^n}{e^n}\right] \|v(\tau)\|_U + \left[\frac{M_{\xi-\eta}}{e} + c\frac{M_{\xi-\eta}^2}{e^2} + \ldots + c^{n-1} \frac{M_{\xi-\eta}^n}{e^n}\right] \\
&\quad\quad \times g(\tau) + \tau^{1-\gamma} \left[c\frac{M_{\xi-\eta}}{e} + c^2 \frac{M_{\xi-\eta}^2}{e^2} + \cdots + c^n \frac{M_{\xi-\eta}^n}{e^n}\right] \|w(\tau)\|_\eta \\
&\quad \leq \frac{1}{1 - \frac{cM_{\xi-\eta}}{e}} \|v(\tau)\|_U + \frac{\frac{cM_{\xi-\eta}}{e}}{1 - \frac{cM_{\xi-\eta}}{e}} \tau^{1-\gamma} \|w(\tau)\|_\eta + \frac{\frac{M_{\xi-\eta}}{e}}{1 - \frac{cM_{\xi-\eta}}{e}} g(\tau) := G(\tau) \ \ \text{(say)}.
\end{aligned}$$

Therefore, $G \in L^p([0, b], [0, \infty))$ and consequently, by vector-valued dominated convergence theorem, we can conclude that $u \in L^p([0, b], U)$.

Next, since w is a mild solution of (4.1), it satisfies

$$w(\tau) = T_{\zeta,\delta}(\tau) y_0 + \int_0^\tau (\tau - \phi)^{\delta-1} R_\delta(\tau - \phi) Bv(\phi) d\phi, \quad \tau \in (0, b].$$

Now, consider the following semilinear system:

$$\left.\begin{aligned}
D_{0^+}^{\delta,\zeta} y(\tau) =& - Ay(\tau) + f(\tau, y(\tau), u(\tau)) + Bv(\tau) - f(\tau, w(\tau), u(\tau)), \\
& \qquad \tau \in (0, b], \\
I_{0^+}^{(1-\delta)(1-\zeta)} w(0) =& y_0.
\end{aligned}\right\} \tag{4.2}$$

Then the mild solution of (4.2) satisfies

$$y(\tau) = T_{\zeta,\delta}(\tau)y_0 + \int_0^\tau (\tau-\phi)^{\delta-1}R_\delta(\tau-\phi)\big[f(\phi,y(\phi),u(\phi)) + Bv(\phi) - f(\phi,w(\phi),u(\phi))\big]d\phi, \quad \tau\in(0,b].$$

Now, for $\tau\in(0,b]$, we have

$$\begin{aligned}
&\tau^{1-\gamma}\|y(\tau)-w(\tau)\|_\eta \\
&\quad\leq \tau^{1-\gamma}\int_0^\tau (\tau-\phi)^{\delta-1}\big\|R_\delta(\tau-\phi)\big[f(\phi,y(\phi),u(\phi)) - f(\phi,w(\phi),u(\phi))\big]\big\|_\eta d\phi \\
&\quad\leq \frac{NM_Q M_{\xi-\eta}\tau^{1-\gamma}}{\Gamma(\delta)}\int_0^\tau (\tau-\phi)^{\delta-1}\phi^{1-\gamma}\|y(\phi)-w(\phi)\|_\eta d\phi.
\end{aligned}$$

Take $H(t) = t^{1-\gamma}\|y(t)-w(t)\|_\eta$. Then, from the above inequality, we have

$$H(\tau) \leq \frac{NM_Q M_{\xi-\eta}\tau^{1-\gamma}}{\Gamma(\delta)}\int_0^\tau (\tau-\phi)^{\delta-1}H(\phi)d\phi, \quad \tau\in[0,b].$$

Using Lemma 2, we get $y = w$, that is, every solution of (4.1) with control v is a solution of the semilinear system (1.1) with control u. Hence, $R_b(0)\subset R_b(f)$, and therefore $\overline{R_b(f)} = Y_\eta$.

Subsequently, problem (1.1) is approximately controllable.

Remark: When $\zeta = 0$, (1.1) reduces to

$$\begin{aligned}
{}^{RL}D^\delta_{0+}y(\tau) &= -Ay(\tau) + f(\tau,y(\tau),u(\tau)) + Bu(\tau), \quad \tau\in(0,b], \\
I^{1-\delta}_{0+}y(0) &= y_0,
\end{aligned}$$

and it can be easily seen that the results in Theorems 4 and 5 hold for $\zeta = 0$ when γ is replaced by δ.

5. Conclusion

In this work, we discuss the approximate controllability of a Hilfer fractional differential equation with control in the nonlinear term. The existence and uniqueness result is proved with the help of a fixed point theorem by utilizing the properties of the fractional powers of A^η, the semigroup $\{Q(\tau)\}_{\tau\geq 0}$ and the associated operators $\{R_\delta(\tau)\}_{\tau>0}$, $\{S_\delta(\tau)\}_{\tau>0}$ and $\{T_{\zeta,\delta}(\tau)\}_{\tau>0}$. For the approximate controllability result, based on the information available, we construct a sequence of functions belonging to the space of admissible controls that converges to a control function $u\in L^p([0,b],U)$, which, by using the definition of mild solution and reachable sets, gives the approximate controllability of our system.

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